# MATCHING OF HECKE OPERATORS FOR EXCEPTIONAL DUAL PAIR CORRESPONDENCES

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*to Steve Rallis, in memoriam*

Abstract. Let $\mathbf{G}$ be a split algebrac group of type $E_n$ defined over a $p$-adic field. This group contains a dual pair $G \times G'$ where one of the groups is of type $G_2$. The minimal representation of $\mathbf{G}$, when restricted to the dual pair, gives a correspondence of representations of the two groups in the dual pair. We prove a matching of spherical Hecke algebras of $G$ and $G'$, when acting on the minimal representation. This implies that the correspondence is functorial, in the sense of Arthur and Langlands, for spherical representations.

## 1. Introduction

Let $F$ be a $p$-adic field with ring of integers $\mathcal{O}$ and fixed inverse of uniformizer $\varpi$. Let $q$ be the order of the residual field. We fix an absolute value on $F$ so that $|\varpi| = q$. We consider exceptional dual pairs $G \times G'$ inside of an adjoint group $\mathbf{G}$ where each group consists of the $F$ points of a split reductive algebraic group. Denote by $K$ and $K'$ hyperspecial maximal compact subgroups of $G$ and $G'$ respectively. Let $(\Pi, \mathbf{V})$ be the minimal representation of $\mathbf{G}$.

If $\sigma'$ is an irreducible representation of $G'$, we call an irreducible representation $\sigma$ of $G$ a $\Theta$-*lift of* $\sigma'$ if $\sigma \otimes \sigma'$ is a quotient of $\Pi$. If $\sigma, \sigma'$ are spherical, we will prove that the correspondence $\sigma \leftrightarrow \sigma'$ is functorial with respect to a natural injection on the dual groups

$$r : \widehat{G}'(\mathbb{C}) \to \widehat{G}(\mathbb{C}).$$

To be more precise, let $C$ be the centralizer of $r(\widehat{G}')$ in $\widehat{G}$. Then $C$ is a reductive, possibly finite, group. Let $f : \mathrm{SL}_2(\mathbb{C}) \to C$ be a map corresponding to the regular unipotent orbit in $C$ by Jacobson-Morozov. Let

$$s = f \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Let $\mathcal{H}$ and $\mathcal{H}'$ denote the spherical Hecke algebras of $G$ and $G'$ respectively. Let $T \in \mathcal{H}$ correspond, via the Satake isomorphism, to a finite dimensional representation $V$ of $\widehat{G}(\mathbb{C})$. (We describe this in more detail in Section 2 below.) Write $V = \sum V' \otimes V''$, the restriction of $V$ to $\widehat{G}' \otimes C$. We define a map $\tilde{r} : \mathcal{H} \to \mathcal{H}'$ by

$$\tilde{r}(V) = \sum_{V'} \mathrm{Tr}_{V''}(s) V'. \tag{1}$$

Note that if $C$ is finite, then $s = 1$, the identity in $\widehat{G}$, and $\tilde{r}(V)$ is just the restriction of $V$ to $\widehat{G}'$. We consider the following dual pairs:

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| $\mathbf{G}$ | $D_4$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| $G'$ | $S_3$ | $\mathrm{PGL}_2$ | $\mathrm{PGL}_3$ | $G_2$ | $G_2$ |
| $G$ | $G_2$ | $G_2$ | $G_2$ | $\mathrm{PGSp}_6$ | $F_4$ |

Table 1. Dual pairs $G \times G' \subset \mathbf{G}$

**Theorem 1.1.** *For the dual pairs $G \times G' \subset \mathbf{G}$ in Table 1, $T \in \mathcal{H}_G$ and $\widetilde{r}$ given by* (1), *$\Pi(T) = \Pi(\widetilde{r}(T))$ as operators on $\mathbf{V}^{K \times K'}$.*

As a matter of terminology, if the actions of $T$ and $\widetilde{r}(T)$ agree on a space $V$, or, more precisely, on a subset of fixed vectors, for all $T \in \mathcal{H}_G$ we will say there is a *matching of Hecke operators* of $\mathcal{H}_G$ and $\mathcal{H}_{G'}$ on $V$ or, more concisely, *matching* on $V$. We trust that the precise space of fixed vectors will be clear from context.

An analogue of Theorem 1.1 is well known in the case of classical theta correspondences [5]. For exceptional groups, the first example of matching was obtained by Rallis and Soudry in [6].

The proof of Theorem 1.1 is by induction on the rank of $\mathbf{G}$. The main tool is Jacquet functors of $\mathbf{V}$ with respect to maximal parabolic subgroups of dual pairs. Most of the needed functors were computed in [4]. One remaining, but rather remarkable case (for $\mathbf{G} = E_8$), is computed in the last section.

## 2. The Satake isomorphism

Before giving the proof of Theorem 1.1 we review the facts about the Hecke algebra and the Satake isomorphism (most of which can be found in [2]) which will be relevant, and we give some general lemmas which will be key in the proof of Theorem 1.1.

2.1. **The Hecke algebra.** Let $G$ be a split reductive group, $K$ a hyperspecial maximal compact subgroup, $B = TU$ a Borel subgroup. There is an Iwasawa decomposition $G = BK$, and the choice of Borel gives a set $\Phi^+$ of positive roots.

We identify the cocharacters of $T$, $\mathrm{X}_\bullet(T)$, with the coweight lattice $\Lambda_c$ so that for every cocharacter $\lambda : F^\times \to T$, the adjoint action of $\lambda(t)$ on the root subgroup of $U$, corresponding to a root $\alpha$, is given by the multiplication by the scalar $t^{\langle \lambda, \alpha \rangle}$. We have a Cartan decomposition:

**Proposition 2.1.** *The group $G$ is the disjoint union of double cosets $K\lambda(\varpi)K$ for $\lambda \in \Lambda_c^+$ where*

$$\Lambda_c^+ = \{\lambda \in \Lambda_c \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

*Example.* $G = \mathrm{GL}_3(F)$, $\widehat{G} = \mathrm{GL}_3(\mathbb{C})$. If $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$ are the simple roots, then $P^+$ consists of $\lambda = (l, m, n)$ such that $l \geq m \geq n$. Then

$$\lambda(t) = \begin{pmatrix} t^l & & \\ & t^m & \\ & & t^n \end{pmatrix}$$

and it is easy to verify directly, from the theory of elementary divisors, that

$$\mathrm{GL}_3(F) = \bigsqcup_{l \geq m \geq n} \mathrm{GL}_3(\mathcal{O}) \begin{pmatrix} \varpi^l & & \\ & \varpi^m & \\ & & \varpi^n \end{pmatrix} \mathrm{GL}_3(\mathcal{O}).$$

Irreducible representations of $\widehat{G}$, the complex dual group of $G$, are parameterized by their highest weights $\lambda \in \Lambda_c^+$. Let $R(\widehat{G})$ be the representation ring of $\widehat{G}$. That is, $R(\widehat{G})$ is the $\mathbb{C}$-vector space with basis consisting of the irreducible representations of $\widehat{G}$. We denote the representation of highest weight $\lambda$ by $V_\lambda$, and consider the map

$$R(\widehat{G}) \to \mathbb{C}[\Lambda_c] \qquad V_\mu \mapsto \sum_\mu m_\lambda(\mu)[\mu] \tag{2}$$

where $m_\lambda(\mu)$ is the dimension of the $\mu$-weight space in $V_\lambda$. Letting $W$ denote the Weyl group, this gives an isomorphism $R(\widehat{G}) \simeq \mathbb{C}[\Lambda_c]^W$.

The Hecke algebra $\mathcal{H}_G$ consists of all locally constant compactly supported $K$-biinvariant functions $f : G \to \mathbb{C}$. By Proposition 2.1, $\mathcal{H}_G$ has a basis consisting of the characteristic functions of $K\lambda(\varpi)K$ with $\lambda \in \Lambda_c^+$. We denote these by $T_\lambda$.

If $(\sigma, V)$ is a smooth $G$-module then the action of $f \in \mathcal{H}_G$ is given by

$$f * v = \int_G f(g)\sigma(g)v dg.$$

We normalize the Haar measure $dg$ so that $\mathrm{vol}(K) = 1$. Let $U$ be the unipotent radical of $B$ as above. By the Iwasawa decomposition, we can write

$$K\lambda(\varpi)K = \bigcup_u \bigcup_t utK \tag{3}$$

for some representatives $t \in T$ and $u \in U$. So if $v$ is a $K$-fixed vector then

$$T_\lambda * v = \sum_{u,t} \sigma(u)\sigma(t)v. \tag{4}$$

If $r_U : V \to V_U$ is the natural projection, where $V_U$ is the space of $U$-coinvariants, then

$$r_U(T_\lambda * v) = \sum_t n(t) r_U(\sigma(t)v) \tag{5}$$

where $n(t)$ is the number of single cosets of type $utK$ appearing in $K\lambda(\varpi)K$.

2.2. **The (relative) Satake transform.** Let $\delta_U$ denote the modular character of $B$ given by

$$d(bub^{-1}) = \delta_U(b)du.$$

The measure $du$ is normalized so that $\mathrm{vol}(K \cap U) = 1$. Obviously, $\delta_U$ is trivial on $U$, and so this defines a character $\delta : T \to \mathbb{R}_+^\times$. We take $\delta_U^{1/2}(t)$ to be the positive square root of this character. Let $\Phi^+$ denote the positive roots of $G$ (determined by $B$.) Then if $\mu \in X_\bullet(T)$,

$$\delta^{1/2}(\mu(\varpi)) = q^{\langle \mu, \rho \rangle}, \qquad 2\rho = \sum_{\alpha \in \Phi^+} \alpha. \tag{6}$$

The Satake transform $\mathcal{S}_T : \mathcal{H}_G \to \mathcal{H}_T$ is given by

$$\mathcal{S}_T f(t) = \delta(t)^{1/2} \int_U f(tu)du. \tag{7}$$

It is a fact that $\mathcal{S}_T$ is injective and its image is equal to the Weyl group invariants. Since $\mathcal{H}_T = \mathbb{C}[\mathrm{X}_\bullet(T)] = \mathbb{C}[\Lambda_c]$, it follows from our discussion above that this defines an isomorphism $\mathcal{S} : \mathcal{H}_G \to R(\widehat{G})$.

The Satake transform can be defined analogously for any parabolic $P = MN$. This yields the *relative Satake transform* $\mathcal{S}_M : \mathcal{H}_G \to \mathcal{H}_M$:

$$\mathcal{S}_M f(m) = \delta_N^{1/2}(m) \int_N f(mn) dn. \tag{8}$$

As above, $dn$ is the measure which gives $N \cap K$ volume 1, and $\delta_N : M \to \mathbb{R}_+^\times$ is the modular character.

We may assume $P \supset B$, so the composition of $\mathcal{S}_M$ with the Satake transform from $\mathcal{H}_M$ to $\mathcal{H}_T$ is $\mathcal{S}_T$.

2.3. **Some general lemmas.** In this section, we prove various simple lemmas which will be used in our proof of Theorem 1.1.

Throughout this paper parabolic induction and the Jacquet functors will be normalized as follows. Let $P = MN$ be a parabolic subgroup of $G$. Suppose that $(\sigma, W)$ is a representation of $M$ which we extend trivially to $P$. Then we define $(\rho, i_P^G(W))$ to be the representation of $G$ consisting of smooth functions $f : G \to W$ which satisfy,

$$f(mng) = \delta_N^{1/2}(m)\sigma(m)f(g) \qquad \text{for all } m \in M, n \in N, g \in G \tag{9}$$

with right regular action $\rho(g)f : h \mapsto f(hg)$. Note that the usual induction functor is

$$\mathrm{Ind}_P^G(W) = i_P^G(\delta_N^{-1/2} \otimes W). \tag{10}$$

If $(\pi, V)$ is a representation of $G$, the Jacquet functor with respect to $N$, $(\pi_M, r_N(V))$, is defined as follows. As usual,

$$V(N) = \langle \pi(n)v - v \mid n \in N, v \in V \rangle,$$

so $V_N = V/V(N)$ is the space of coinvariants. Let $(\pi_M, r_N(V))$ be a representation of $M$ such that $r_N(V) = V_N$, but the $M$-action is given by

$$\pi_M(m)v = \delta_N^{-1/2}(m)\pi(m)v. \tag{11}$$

Since $N$ is normal, it is trivial to see that this is well defined.

Since the induction and the Jacquet functor are normalized, the statement of Frobenius reciprocity is quite simple:

$$\mathrm{Hom}_G(V, i_P^G(W)) = \mathrm{Hom}_M(r_N(V), W). \tag{12}$$

**Lemma 2.2.** *Suppose that $G$ is a split reductive group, $K \subset G$ is a maximal compact subgroup and $MN = P \subset G$ is a parabolic subgroup. Let $K_M = K \cap M$. For $(\pi, V)$ any smooth representation of $M$, the following statements hold.*

(i) *The map $\varphi : (i_P^G(V))^K \to V^{K_M}$ given by $f \mapsto f(1)$ is an isomorphism.*
(ii) *If $T \in \mathcal{H}$ and $v \in (i_P^G(V))^K$ then $\varphi(T * v) = \mathcal{S}_M(T) * \varphi(v)$.*

*Proof.* Since $G = PK$ any element $g$ in $G$ can be written as $g = mnk$, where $k \in K$, $mn \in P$, and we can define

$$\psi : V^{K_M} \to (i_P^G(V))^K$$

by specifying that $\psi(v)(mnk) = \delta_N(m)^{1/2}\pi(m)v$. This is well defined precisely because $v \in V^{K_M}$. Now, (i) follows by computing that $\psi$ is the inverse of $\varphi$.

For remainder of the proof let $\mathcal{S} = \mathcal{S}_M$. Let $T_\lambda$ be the characteristic function of $K\lambda(\varpi)K$. Note that $\mathcal{S}(T_\lambda)$ is determined by its values on a set of coset representatives for $M/K_M$. We fix such a set. Using the Iwasawa decomposition $G = PK$, we may write (in analogy to (3))

$$K\lambda(\varpi)K = \bigcup_i \bigcup_j m_i n_i K \tag{13}$$

with the $m_i$ chosen from the given set of coset representatives. If $m \notin K\lambda(\pi)K$, then obviously $\mathcal{S}(T_\lambda)(m) = 0$. Otherwise, $m = m_{j_0}$ for some $m_{j_0}$ appearing in the decomposition (13). Let

$$n(i,j) := \#\{n_i \mid m_j n_i K \text{ appears in (13)}\}.$$

Then

$$\begin{aligned} \mathcal{S}(T_\lambda)(m_{j_0}) &= \delta^{1/2}(m_{j_0}) \int_N T_\lambda(m_{j_0} n) dn \\ &= \delta^{1/2}(m_{j_0}) \sum_{i,j} T_\lambda(m_{j_0} n_i) \\ &= \delta^{1/2}(m_{j_0}) n(i, j_0), \end{aligned}$$

since $\mathrm{vol}(N/(K \cap N)) = 1$.

Let $f \in (\mathrm{Ind}_P^G V)^K$. By (i), $\varphi(f) = f(1) = v \in V^{K_M}$. So, by the previous calculation,

$$\begin{aligned} \mathcal{S}(T_\lambda) * v &= \int_M \mathcal{S}(T_\lambda)(m)\pi(m) v dm \\ &= \sum_j \mathcal{S}(T_\lambda)(m_j)\pi(m_j) v \\ &= \sum_j \delta^{1/2}(m_j) n(i,j)\pi(m_j) v. \end{aligned}$$

On the other hand, since $f$ is fixed by $K$,

$$\begin{aligned} \varphi(T_\lambda * f) = (T_\lambda * f)(1) &= \int_G T_\lambda(g)\sigma(g) f(1) dg \\ &= \sum_{i,j} f(1 m_j n_i) \\ &= \sum_j n(i,j)\delta^{1/2}(m_j)\pi(m_j) v. \end{aligned}$$

By Proposition 2.1, $\{T_\lambda \mid \lambda \in \Lambda_c^+\}$ forms a basis of $\mathcal{H}$. Therefore, (ii) is proved. □

**Lemma 2.3.** *Let $G$ be a reductive group with $P = MN$ the Levi decomposition of a parabolic. If $V$ is any $G$-module, the map $V^K \to V_N^{K_M}$ is injective.*

*Proof.* For $I \subset G$ the Iwahori subgroup, Borel (see [1]) proved that $V^I \hookrightarrow V_N^{I_M}$. As the Jacquet functor is intertwining for the action of $P$, the image of $V^K$ is clearly fixed by $K_M$. Since $V^K \subset V^I$, this gives the desired result. □

Let $G$ be a split reductive group defined over $F$. If $\chi : G \to \mathrm{GL}_1$ is a character then $\chi^* : \mathbb{C}^\times \to \widehat{G}$ will denote the corresponding co-character. The group $\chi^*(\mathbb{C}^\times)$ is in the center of $\widehat{G}$. For example, if $G = \mathrm{GL}_n$ and $\chi = \det$, then $\chi^*(z)$ is the scalar matrix in $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$.

**Lemma 2.4.** *Let $\chi : G \to \mathrm{GL}_1$ be a character. Let $V$ be a finite dimensional irreducible representation of $\widehat{G}$ (i.e. a Hecke operator for $G$). Let $m$ be a half integer. Then $\chi^*(q^m)$ acts on $V$ as $q^n$ for some half integer $n$. Let $\pi$ be a representation of $G$. Then $V$ acts on $\pi \otimes |\chi|^m$ as $q^n \cdot V$ acts on $\pi$.*

**Lemma 2.5.** *Let $\pi$ be a representation of $G = G' \times G''$ obtained as the pullback of $\pi'$, a representation of $G'$. Let $s \in \widehat{G}''$ be the image of $\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right)$ under the principal $\mathrm{SL}_2 \to \widehat{G}''$. For $V$ a finite dimensional representation of $\widehat{G}$ (i.e. a Hecke operator for $G$), write $V = \sum V' \otimes V''$, the restriction of $V$ to $\widehat{G}' \times \widehat{G}''$. Then $V$ acts on $\pi$ as $\sum \mathrm{Tr}_{V''}(s) V'$ acts on $\pi'$.*

*Proof.* Since the Satake parameter of the trivial representation of a group $G''$ is $s$, the result is clear. $\square$

**Lemma 2.6.** *Let $\chi : G \to \mathrm{GL}_1$ be a character, and let $C = \chi^*(\mathbb{C}^\times) \subseteq \widehat{G}$. Let $\pi'$ be a representation of $\mathrm{GL}_1$. Then $\pi = \pi' \circ \chi$ is a representation of $G$. Let $s \in \widehat{G}$ be the image of $\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right)$ under the principal $\mathrm{SL}_2 \to \widehat{G}$. For $V$ a finite dimensional representation of $\widehat{G}$ (i.e. a Hecke operator for $G$), write $V = \sum V' \otimes V''$, the restriction of $V$ to $C \times \mathrm{SL}_2$. Then $V$ acts on $\pi$ as $\sum \mathrm{Tr}_{V''}(s) V'$ acts on $\pi'$.*

*Proof.* We prove this for $G = \mathrm{GL}_n$ and $\chi = \det$ is the determinant. In this case $C$ is the center. It suffices to prove the statement for the fundamental representations $V_{\lambda_i} = V_i = \wedge^i \mathbb{C}^n$ where

$$\lambda_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0).$$

Since $V_i$ is miniscule, the Satake isomorphism gives $\mathcal{S}(T_{\lambda_i}) = q^{i(n-i)/2} V_i$. Therefore, the action of $\pi(V_i)$ is

$$\begin{aligned} q^{i(i-n)/2} \int_{K\lambda_i(\varpi)K} \pi(g)dg =& q^{i(i-n)/2} \operatorname{vol}(K\lambda_i(\varpi)K) \pi'(\det(\lambda_i(\varpi))) \\ =& q^{i(i-n)/2} \operatorname{vol}(K\lambda_i(\varpi)K) \pi'(\varpi)^i. \end{aligned}$$

The center $C$ acts on $V_i$ by the character $z \mapsto z^i$. Note that

$$s = \begin{pmatrix} q^{(n-1)/2} & & & \\ & q^{(n-3)/2} & & \\ & & \ddots & \\ & & & q^{(1-n)/2} \end{pmatrix},$$

is the image of $\left(\begin{smallmatrix} q^{1/2} & \\ & q^{-1/2} \end{smallmatrix}\right)$ under the principal $\mathrm{SL}_2 \to \widehat{G}$. So, to complete the proof we just need to show that

$$q^{i(i-n)/2} \operatorname{vol}(K\lambda_i(\varpi)K) = \mathrm{Tr}_{V_i}(s).$$

This is immediate from the discussion in [2, Section 3] and the fact that $V_i$ is miniscule. $\square$

**Lemma 2.7.** *Let $G$ be a reductive group. Let $C_c^\infty(G)$ denote the space of smooth, compactly supported functions on $G$. This is a $G \times G$ module for the left and right action of $G$ called the* regular representation. *On $C_c^\infty(G)^{K \times K}$ we have a matching of the Hecke algebras for the left and right action.*

*Proof.* This is obvious since $C_c^\infty(G)^{K\times K}$ is nothing else but the Hecke algebra itself. To be precise, since the left action on $f$ in $C_c^\infty(G)$ is by $\lambda_g(f)(x) = f(g^{-1}x)$, a Hecke operator $R$, acting from the right, is matched with $R^*$ defined by $R^*(x) = R(x^{-1})$. $\square$

*Remark.* Notice that this matching of Hecke operators, when considered as a matching of virtual representations, matches $V \in R(G)$ with its dual $\widetilde{V}$. In particular, if $V$ is self-dual then it is matched with itself.

## 3. Our groups

3.1. **Octonions.** Let $\mathbb{O}$ denote the non-associative division algebra of rank 8 over $F$. There is an $F$-linear anti-involution $x \mapsto \bar{x}$ on $\mathbb{O}$, hence norm and trace maps

$$\mathbb{N} : \mathbb{O} \to F \quad x \mapsto x\bar{x} = \bar{x}x, \qquad \mathrm{Tr} : \mathbb{O} \to F, \quad x \mapsto x + \bar{x}$$

satisfying

$$\mathbb{N}(x \cdot y) = \mathbb{N}(x)\mathbb{N}(y), \qquad \mathrm{Tr}(x \cdot y) = \mathrm{Tr}(y \cdot x), \quad \mathrm{Tr}(x \cdot (y \cdot z)) = \mathrm{Tr}((x \cdot y) \cdot z).$$

On the set $\mathbb{O}^0$ of trace zero elements, we have $\bar{x} = -x$. The group $G_2$ is the automorphism group of $\mathbb{O}$.

The quadratic form $\mathbb{N} : \mathbb{O} \to F$ has signature $(4, 4)$, which means that $\mathbb{O}$ has a basis $\{1, i, j, k, l, li, lj, lk\}$. (Note that $l^2 = 1$.) The following basis is particularly useful.

$$\begin{array}{llll} s_1 = \frac{1}{2}(i + li), & s_2 = \frac{1}{2}(j + lj), & s_3 = \frac{1}{2}(k + lk), & s_4 = \frac{1}{2}(1 + l), \\ t_1 = \frac{1}{2}(i - li), & t_2 = \frac{1}{2}(j - lj), & t_3 = \frac{1}{2}(k - lk), & t_4 = \frac{1}{2}(1 - l). \end{array} \tag{14}$$

The multiplication table for this basis is given in Table 2.

| | $s_1$ | $s_2$ | $s_3$ | $t_1$ | $t_2$ | $t_3$ | $s_4$ | $t_4$ |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | $-t_3$ | $t_2$ | $s_4$ | 0 | 0 | 0 | $s_1$ |
| $s_2$ | $t_3$ | 0 | $-t_1$ | 0 | $s_4$ | 0 | 0 | $s_2$ |
| $s_3$ | $-t_2$ | $t_1$ | 0 | 0 | 0 | $s_4$ | 0 | $s_3$ |
| $t_1$ | $t_4$ | 0 | 0 | 0 | $s_3$ | $-s_2$ | $t_1$ | 0 |
| $t_2$ | 0 | $t_4$ | 0 | $-s_3$ | 0 | $s_1$ | $t_2$ | 0 |
| $t_3$ | 0 | 0 | $t_4$ | $s_2$ | $-s_1$ | 0 | $t_3$ | 0 |
| $s_4$ | $s_1$ | $s_2$ | $s_3$ | 0 | 0 | 0 | $s_4$ | 0 |
| $t_4$ | 0 | 0 | 0 | $t_1$ | $t_2$ | $t_3$ | 0 | $t_4$ |

Table 2. Multiplication Table for Octonions

*Remark.* From this basis it is evident that a subspace $V \subset \mathbb{O}^0$ on which multiplication is trivial is at most 2-dimensional. (We call such a subspace a *null space* or a *null subspace.*) Indeed, let $\{i, j, k\} = \{1, 2, 3\}$. Then from the multiplication table we see that $s_i^\perp = \langle s_i, t_j, t_k\rangle$, and the null spaces of $\mathbb{O}^0$ which contain $s_i$ are all of the form $\langle s_i, at_j + bt_k\rangle$ for fixed $a, b \in F$. Since $G_2$ acts transitively on (nonzero) elements of trace zero and norm zero, this phenomenon is generic.

3.1.1. *Maximal parabolic subgroups in $G_2$.* They are described as the stabilizers of null subspaces $V \subset \mathbb{O}^0$. Let $V_1$ be spanned by $s_1$ and $V_2$ by $s_1$ and $t_2$. Then $V_3 = V_1^{\perp}$ is spanned by $s_1, t_2$ and $t_3$. Let $P_1 = M_1N_1$ and $P_2 = M_2N_2$ be the stabilizers of $V_1$ and $V_2$, respectively. The Levi factor $M_2$ acts on $V_2$. The choice of the basis in $V_2$ gives an isomorphism $M_2 \cong \mathrm{GL}_2$. The Levi factor $M_2$ acts on $V_3/V_1$ and we have an isomorphism $M_1 \cong \mathrm{GL}_2$. It is not difficult to see that $g \in \mathrm{GL}_2 \cong M_1$ acts on $V_1$ by $\det(g)$. The set of all $g$ in $G_2$ such that all $s_i$ and $t_i$ are eigenvectors is a maximal split torus $T$ in $G_2$. The modular characters are

$$\delta_{N_1}(g) = |\det(g)|^5 \text{ and } \delta_{N_2}(g) = |\det(g)|^3. \tag{15}$$

The stabilizer of $s_4 - t_4$ in $G_2$ is a group isomorphic to $\mathrm{SL}_3$. Under the action of this group we have a decomposition

$$\mathbb{O}^0 = \langle s_1, s_2, s_3 \rangle \oplus \langle t_1, t_2, t_3 \rangle \oplus \langle s_4 - t_4 \rangle.$$

We can identify the stabilizer of $s_4 - t_4$ with $\mathrm{SL}_3$ so that the action on $\langle s_1, s_2, s_3 \rangle$ is standard. The torus $T$ of $G_2$ sits in $\mathrm{SL}_3$. In this way, we can represent elements in $T$ by $3 \times 3$ matrices. For example, if $\alpha_l$ is a long root and $\alpha_s$ a short root perpendicular to $\alpha_l$ then, up to Weyl group conjugation,

$$\alpha_l^*(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \text{ and } \alpha_s^*(t) = \begin{pmatrix} t & & \\ & t^{-2} & \\ & & t \end{pmatrix}. \tag{16}$$

3.2. **Description of groups.** Let $P = MN$ be a maximal parabolic subgroup of $\mathbf{G}$ as in the table below. The group $N$ is abelian, except in the case $E_8$, where $N$ has one-dimensional center $Z$. In order to give a uniform notation, let $Z$ be trivial if $N$ is abelian. Let $d$ denote the dimension of $N/Z$. Let $C \cong \mathrm{GL}_1$ be the center of $M$. Fix an isomorphism $\lambda_* : \mathrm{GL}_1 \to C$ such that the adjoint action of $\lambda_*(z)$ on $N/Z$ is given by multiplication by $z$. We have a dual pair $G_2 \times H \subset \mathbf{G}$ such that $Q = LU = H \cap P$ is a maximal parabolic of $H$. Let $N_0 \subseteq N/Z$ be the complement of $\bar{U}/\bar{Z}$ under the invariant pairing induced by the Killing form. We fix an isomorphism of $L$ with a classical group so that the action of $G_2 \times L$ on $N_0$ is isomorphic to $\mathbb{O}_0 \otimes F^n$, the space of $n$-tuples in $\mathbb{O}_0$, and $h \in L$ acts on an $n$-tuple $(x_1, \ldots, x_n)$ by $(x_1, \ldots, x_n)h^{-1}$. Since the scalar matrix $z^{-1}$ in $L$ acts on $N/Z$ as $z$, the center of $L$ coincides with the center of $M$. In the case of $\mathbf{G} = E_8$, let $i$ be the isogeny character of $\mathrm{GSp}_6$.

| $\mathbf{G}$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|
| $M$ | $D_4$ | $D_5$ | $E_6$ | $E_7$ |
| $d$ | 8 | 16 | 27 | 56 |
| $L$ | $\mathrm{GL}_1$ | $\mathrm{GL}_2$ | $\mathrm{GL}_3$ | $\mathrm{GSp}_6$ |
| $\delta_{\bar{U}}$ | $\lvert\det\rvert$ | $\lvert\det\rvert$ | $\lvert\det\rvert^2$ | $\lvert i\rvert^8$ |
| $\delta_{\bar{N}}$ | $\lvert\det\rvert^8$ | $\lvert\det\rvert^8$ | $\lvert\det\rvert^9$ | $\lvert i\rvert^{29}$ |

TABLE 3. Maximal parabolic subgroups

The last row of the table is the restriction of the character $\delta_{\bar{N}}$ to $L$. The group $\mathrm{GSp}_6$ acts by the isogeny character $i$ on $\bar{Z}$.

3.2.1. *Maximal parabolic subgroups in $L$.* Assume first that $L = \mathrm{GL}_n$. Recall that $g \in \mathrm{GL}_n$ acts on $F^n$, the space of $n$-tuples $(x_1, \ldots, x_n)$ by $(x_1, \ldots, x_n)g^{-1}$. For $m \leq n$ let $Q_m$ be the stabilizer of the subspace consisting of the $n$-tuples whose last $n-m$ entries are 0. We have a Levi decomposition $Q_m = L_m U_m$ where $L_m = \mathrm{GL}_m \times \mathrm{GL}_{n-m}$ is the stabilizer, in $Q_m$, of the subspace consisting of the $n$-tuples whose first $m$ entries are 0. Let $g = (g_1, g_2) \in \mathrm{GL}_{n-m} \times \mathrm{GL}_m$. The modular character $\delta_{U_m}$ is

$$\delta_{U_m}(g) = |\det(g_1)|^{m-n} \cdot |\det(g_2)|^m. \tag{17}$$

Assume now that $L = \mathrm{GSp}_{2n}$. This is a group of isogenies of a symplectic form $(\cdot,\cdot)$ on a $2n$ dimensional space. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a symplectic basis, i.e.

$$(e_i, f_j) = -(f_j, e_i) = \delta_{ij} \text{ and } (e_i, e_j) = (f_i, f_j) = 0.$$

Using this basis we identify the symplectic space with $F^{2n}$, the space of $2n$-tuples. We identify $\mathrm{GSp}_{2n}$ with the group of $2n \times 2n$ matrices $g$ acting on $F^{2n}$ from the right, and preserving the symplectic form, up to an isogeny character :

$$(vg^{-1}, ug^{-1}) = i(g)^{-1}(v, u).$$

For every $m \leq n$, let $Q_m$ be the subgroup of $\mathrm{GSp}_{2n}$ preserving the subspace spanned by $e_1, \ldots, e_m$. We have a Levi decomposition $Q_m = L_m U_m$ where $L_m$ is the stabilizer, in $Q_m$ of the subspace spanned by $f_1, \ldots, f_m$. Then $Q_m \cong \mathrm{GL}_m \times \mathrm{GSp}_{2(n-m)}$ where $g = (g_1, g_2) \in \mathrm{GL}_m \times \mathrm{GSp}_{2(n-m)}$ acts as follows: $e_i \mapsto e_i g_1^{-1}$, for $1 \leq i \leq m$, $f_i \mapsto i(g_2)^{-1} f_i g_1^{\top}$ for $1 \leq i \leq m$ (here $g_1^{\top}$ is the transpose of $g_1$) and as $g_2^{-1}$ on the remaining $2(n-m)$ basis elements. The modular character $\delta_{U_m}$ is

$$\delta_{U_m}(g) = |\det(g_1)|^{-(2n-m+1)} |i(g_2)|^{\frac{m(2n-m+1)}{2}}. \tag{18}$$

3.2.2. *Example:* $\mathbf{G} = E_7$. Let $J_{27}$ be the 27-dimensional Jordan algebra over $F$ given by

$$J_{27} = \left\{ A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \middle| \, a, b, c \in F, \quad x, y, z \in \mathbb{O} \right\}. \tag{19}$$

The determinant on $J_{27}$ gives an $F$-valued cubic form on $J_{27}$. The adjoint group $\mathbf{G}$ has a maximal parabolic $P = MN$ such that

$$M \cong \{g \in \mathrm{GL}(J_{27}) \mid \det(g(A)) = \lambda(g)\det(A) \text{ for some similitude } \lambda(g) \in F^{\times}\}, \tag{20}$$

a reductive group of type $E_6$, and $N \cong J_{27}$ as $M$-modules. Moreover, $\bar{N} \cong J_{27}$, and the natural pairing of $\bar{N}$ and $N$ can be identified with the trace on $J_{27}$.

Evidently, $G_2 \subset M$ acts term by term on the elements of $A$, and since $g \in \mathrm{GL}_3$ acts via

$$g \cdot A = (\det g)^{-1} g A g^t, \tag{21}$$

the action of $G_2$ and $\mathrm{GL}_3$ obviously commute, hence $G_2 \times \mathrm{GL}_3 \subset M$. As described in [4, Section 5], $G_2 \times \mathrm{PGSp}_6 \subset \mathbf{G}$ is a dual pair, and $Q = LU = \mathrm{PGSp}_6 \cap P$ where $U$ can be identified with $J_6$, the subalgebra of $J_{27}$ consisting of (symmetric) matrices with entries in the field $F$, and $L \simeq \mathrm{GL}_3$ with action on $J_{27} \simeq N$ given by (21). Thus, the orthogonal complement $N_0$ of $\bar{U}$ in $N$ is identified with the subspace of $J_{27}$ consisting of matrices with 0 on the diagonal and traceless octonions off the diagonal, that is, $N_0$ is identified with the set of triples $(x, y, z)$ of traceless octonions. Moreover, $(g, h) \in G_2 \times \mathrm{GL}_3$ acts on $(x, y, z)$ by $(gx, gy, gz)h^{-1}$. (The action of $h$ follows from Cramer's rule.)

## 4. The proof

4.1. **The base case.** Let $\mathbf{V}$ be the minimal representation of $D_4$. Let $G' = S_3$ be the group of permutations of 3 letters. Then $S_3$ acts on $D_4$, by outer automorphisms, fixing $G_2$. Since $\mathbf{V}$ can be extended to a representation of a semi-direct product of $D_4$ and $S_3$, we have a dual pair $G \times G' = G_2 \times S_3$ acting on $\mathbf{V}$. We let $\widehat{G}' = S_3$ and

$$r : S_3 \to G_2(\mathbb{C})$$

such that the centralizer $C$ of $r(S_3)$ in $G_2(\mathbb{C})$ is $\mathrm{SO}(3) \subset \mathrm{SL}_3(\mathbb{C}) \subset G_2(\mathbb{C})$. (The group $\mathrm{SO}(3) \simeq \mathrm{PGL}_2(\mathbb{C})$ corresponds to the subregular unipotent orbit by the Jacobson-Morozov theorem.)

Let $K' = G'$. Then the Hecke algebra $\mathcal{H}'$ is one-dimensional. With these choices, Theorem 1.1 asserts that the $S_3$-invariants of the minimal representation of $D_4$ is the unramified representation $\pi_{sr}$ of $G_2$ whose Satake parameter corresponds to the subregular orbit. This is proved in [3].

4.2. **The general case.** Assume that $\mathbf{G} \neq D_4$. Then we have a maximal parabolic $P = MN$ in $\mathbf{G}$ and the corresponding maximal parabolic $Q = LU$ in $H$ as in Table 3. For simplicity, let $K$ be the maximal compact subgroup of $G_2$, and $K'$ that of $H$. Assume that we want to show matching of two operators $T$ and $T'$. By Lemma 2.3,

$$\mathbf{V}^{K \times K'} \hookrightarrow r_{\bar{U}}(\mathbf{V})^{K \times K'_L}$$

where $K'_L = K' \cap L$. Thus Theorem 1.1 holds if we can show matching of $T$ and $\mathcal{S}_L(T')$ on $r_{\bar{U}}(\mathbf{V})^{K \times K'_L}$. If $\mathbf{G} \neq E_8$, the unnormalized Jacquet functor $\mathbf{V}_{\bar{U}}$ was computed in [4]. In the context of the present work, we find it convenient to describe these results in terms of maximal parabolic subgroups $Q_m = L_m U_m$ of $L \cong \mathrm{GL}_n$ and maximal parabolic subgroups $P_m = M_m N_m$ of $G_2$ (as defined in Sections 3.2.1 and 3.1.1, respectively). In particular, we have fixed isomorphisms $L_m \cong \mathrm{GL}_m \times \mathrm{GL}_{n-m}$ and $M_m \cong \mathrm{GL}_2$.

Let $s, t$ be a pair of real numbers. For $m = 1, 2$, let $C_c^\infty(\mathrm{GL}_m)[s,t]$ be the vector space $C_c^\infty(\mathrm{GL}_m)$ with an $M_m \times L_m$-module structure defined by

$$(g_1, g_2) \cdot f(h) = |\det g_1|^s \, |\det g_2|^t \, f(hg_1)$$

for any $(g_1, g_2) \in L_m$, and by

$$g \cdot f(h) = \begin{cases} f(\det g^{-1} h) & \text{if } m = 1, \\ f(g^{-1} h) & \text{if } m = 2, \end{cases} \tag{22}$$

for any $g \in M_m$.

We shall omit $[s,t]$ in the notation if $s = t = 0$. With this notation in hand, we now describe the Jacquet module $r_{\bar{U}}(\mathbf{V})$ for each of our cases.

4.2.1. $\mathbf{G} = D_5$.

**Proposition 4.1.** *As a $G_2 \times \mathrm{GL}_1$-module, $r_{\bar{U}}(\mathbf{V})$ has a filtration with two successive sub quotients*

(1) $i_{P_1}^{G_2}(C_c^\infty(\mathrm{GL}_1))$.
(2) $\mathbf{V}(M) \otimes |\det|^{\frac{1}{2}} \oplus 1 \otimes |\det|^{\frac{5}{2}}$.

*Here $\mathbf{V}(M)$ is the minimal representation of $M/C$.*

*Proof.* This is simply a normalized version of Proposition 2.3 of [4]. Indeed, $\mathbf{V}_{\bar{U}}$ has a filtration with two successive quotients

- $\mathrm{Ind}_{P_1}^{G_2}(C_c^\infty(\mathrm{GL}_1)) \otimes |\det|^3$.
- $\mathbf{V}(M) \otimes |\det| \oplus 1 \otimes |\det|^3$.

where induction is not normalized. Since $r_{\bar{U}}(\mathbf{V}) = \mathbf{V}_{\bar{U}} \otimes \delta_{\bar{U}}^{-\frac{1}{2}}$, and $\delta_{\bar{U}} = |\det|$ by Table 3, $r_{\bar{U}}(\mathbf{V})$ has a filtration with two successive quotients

- $\mathrm{Ind}_{P_1}^{G_2}(C_c^\infty(\mathrm{GL}_1)) \otimes |\det|^{\frac{5}{2}}$.
- $\mathbf{V}(M) \otimes |\det|^{\frac{1}{2}} \oplus 1 \otimes |\det|^{\frac{5}{2}}$.

and (2) follows. Since, for any $m$ and $s$, $C_c^\infty(\mathrm{GL}_m) \cong C_c^\infty(\mathrm{GL}_m) \otimes |\det|^s$ as $\mathrm{GL}_m \times \mathrm{GL}_m$-modules, we can replace $C_c^\infty(\mathrm{GL}_1)$ in the first bullet by $C_c^\infty(\mathrm{GL}_1) \otimes |\det|^{-\frac{5}{2}}$. By (10) and (15), this normalizes the induction for $G_2$ and, at the same time, removes the character $|\det|^{\frac{5}{2}}$ of $\mathrm{GL}_1$. Hence (1) follows. $\square$

*Proof of Theorem 1.1 in case* $\mathbf{G} = D_5$. The dual group of $H = \mathrm{PGL}_2$ is $\mathrm{SL}_2(\mathbb{C})$. The map $r : \mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$ corresponds to a long root $\alpha_l$ of $G_2$: $r(\mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_{2,l}(\mathbb{C}) \subset G_2(\mathbb{C})$. The centralizer $C$ of $\mathrm{SL}_{2,l}(\mathbb{C})$ in $G_2(\mathbb{C})$ is $\mathrm{SL}_{2,s}(\mathbb{C})$ corresponding to a short root $\alpha_s$ perpendicular to $\alpha_l$.

Let $V$ be a finite dimensional representation of $G_2(\mathbb{C})$ and $T_2$ the corresponding Hecke operator for $G_2$. Let

$$s = \alpha_s^*(q^{1/2}) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in \mathrm{SL}_{2,s}(\mathbb{C}). \tag{23}$$

If the restriction of $V$ to $\mathrm{SL}_{2,l}(\mathbb{C}) \times \mathrm{SL}_{2,s}(\mathbb{C})$ is $\sum V' \otimes V''$, we define $T_1$ as corresponding to $\sum \mathrm{Tr}_{V''}(s)V'$. We want to show that $T_2$ matches with $\mathcal{S}_L(T_1)$ on $r_{\bar{U}}(\mathbf{V})$. Since $\widehat{L}$ is the torus $\alpha_l^*(\mathbb{C}^\times) \subseteq \mathrm{SL}_{2,l}(\mathbb{C})$, the operator $\mathcal{S}_L(T_1)$ corresponds to the representation $\sum \mathrm{Tr}_{V''}(s)V'$ of $\alpha_l^*(\mathbb{C}^\times)$ obtained by restricting each $V'$ to the torus $\alpha_l^*(\mathbb{C}^\times)$.

First, we show matching on (1) in Proposition 4.1. By Lemma 2.2, we need to show matching of $\mathcal{S}_{M_1}(T_2)$ and $\mathcal{S}_L(T_1)$ on $C_c^\infty(\mathrm{GL}_1)$. The operator $\mathcal{S}_{M_1}(T_2)$ corresponds to the restriction of $V$ to $\widehat{M_1} \simeq \mathrm{GL}_{2,s}(\mathbb{C})$, the dual group of $M_1$. The center of $\mathrm{GL}_{2,s}(\mathbb{C})$ is the torus $\alpha_l^*(\mathbb{C}^\times) \subseteq \mathrm{SL}_{2,l}(\mathbb{C})$. Let $s$ be as in (23) and let $V = \sum V' \otimes V''$ be the restriction of $V$ to $\mathrm{SL}_{2,l}(\mathbb{C}) \times \mathrm{SL}_{2,s}(\mathbb{C})$, as before. By Lemma 2.6, $\mathcal{S}_{M_1}(T_2)$ acts on $C_c^\infty(\mathrm{GL}_1)$ as the Hecke operator for $\mathrm{GL}_1$ that corresponds to the representation $\sum \mathrm{Tr}_{V''}(s)V'$ of $\alpha_l^*(\mathbb{C}^\times)$, the center of $\mathrm{GL}_{2,s}(\mathbb{C})$. In particular, this is the same $\mathrm{GL}_1$-operator as $\mathcal{S}_L(T_1)$. Matching now follows from Lemma 2.7.

Using the previously proved base case, matching on (2) in Proposition 4.1 reduces to a simple check on two $G_2 \times L$ modules: $\pi_{sr} \otimes |\det|^{\frac{1}{2}}$ and $1 \otimes |\det|^{\frac{5}{2}}$. $\square$

4.2.2. $\mathbf{G} = E_6$.

**Proposition 4.2.** *As a $G_2 \times \mathrm{GL}_2$-module, $r_{\bar{U}}(\mathbf{V})$ has a filtration with three successive sub quotients*

(1) $i_{P_2}^{G_2}(C_c^\infty(\mathrm{GL}_2))$.
(2) $i_{P_1 \times Q_1}^{G_2 \times \mathrm{GL}_2}(C_c^\infty(\mathrm{GL}_1)[-\frac{1}{2}, 1])$.
(3) $\mathbf{V}(M) \otimes |\det|^{\frac{1}{2}} \oplus 1 \otimes |\det|^{\frac{3}{2}}$.

*Here* $\mathbf{V}(M)$ *is the minimal representation of* $M/C$.

This is a normalized version of Theorem 4.3 of [4] which states that $\mathbf{V}_{\bar{U}}$ has a filtration with three successive quotients

- $\mathrm{Ind}_{P_2}^{G_2}(C_c^\infty(\mathrm{GL}_2)) \otimes |\det|^2$.
- $\mathrm{Ind}_{P_1\times Q_1}^{G_2\times \mathrm{GL}_2}(C_c^\infty(\mathrm{GL}_1)) \otimes |\det|^2$.
- $\mathbf{V}(M) \otimes |\det| \oplus 1 \otimes |\det|^2$.

*Proof of Theorem 1.1 in case* $\mathbf{G} = E_6$. The dual group of $H = \mathrm{PGL}_3$ is $\mathrm{SL}_3(\mathbb{C})$. We have an inclusion $r : \mathrm{SL}_3(\mathbb{C}) \to G_2(\mathbb{C})$ where $\mathrm{SL}_3(\mathbb{C}) \subset G_2(\mathbb{C})$ is given by the long roots.

Let $V$ be a finite dimensional representation of $G_2(\mathbb{C})$ and $T_2$ the corresponding Hecke operator for $G_2$. We restrict $V$ to $\mathrm{SL}_3(\mathbb{C})$, and let $T_2$ be the corresponding Hecke operator for $\mathrm{PGL}_3$. We want to show that $T_2$ matches with $\mathcal{S}_L(T_1)$ on $r_{\bar{U}}(\mathbf{V})$.

First, we consider matching on $\mathbf{V}(M) \otimes |\det|^{\frac{1}{2}}$. The dual group of $L = \mathrm{GL}_2$ is $\mathrm{GL}_{2,l}(\mathbb{C}) \subseteq \mathrm{SL}_3(\mathbb{C})$. The group $L$ acts on $\mathbf{V}(M)$ by its quotient $\mathrm{PGL}_2$. The dual group of $\mathrm{PGL}_2$ is $\mathrm{SL}_{2,l}(\mathbb{C})$. Thus, $\mathcal{S}_L(T_1)$ acts on $\mathbf{V}(M)$ as the Hecke operator for $\mathrm{PGL}_2$ that corresponds to the restriction of $V$ to $\mathrm{SL}_{2,l}(\mathbb{C})$. We need to take into account the twist by $|\det|^{\frac{1}{2}}$. Let $\chi$ be the determinant character of $L$. Let $\chi^* : \mathbb{C}^\times \to \mathrm{GL}_{2,l}(\mathbb{C})$ be the corresponding co-character. Note that $\chi^* = \alpha_s^*$. Let $s = \alpha_s^*(q^{1/2}) \in \mathrm{SL}_{2,s}(\mathbb{C})$. Let $V = \sum V' \otimes V''$ be the restriction of $V$ to $\mathrm{SL}_{2,l}(\mathbb{C}) \times \mathrm{SL}_{2,s}(\mathbb{C})$. Let $T$ be the Hecke operator for $\mathrm{PGL}_2$ that corresponds to the representation $\sum \mathrm{Tr}_{V''}(s) V'$ of $\mathrm{SL}_{2,l}(\mathbb{C})$. By Lemma 2.4, $\mathcal{S}_L(T_1)$ acts on $\mathbf{V}(M) \otimes |\det|^{\frac{1}{2}}$ as $T$ acts on $\mathbf{V}(M)$. But $T$ is matched with $T_2$ on $\mathbf{V}(M)$, by the case $\mathbf{G} = D_5$.

To prove matching on (1) in Proposition 4.2 it suffices to show that $\mathcal{S}_{M_2}(T_2)$ and $\mathcal{S}_L(T_1)$ are matching on $C_c^\infty(\mathrm{GL}_2)$, by Lemma 2.2. Since the dual group of $M_2$ is conjugated in $G_2(\mathbb{C})$ to $\mathrm{GL}_{2,l}(\mathbb{C})$, the dual group of $L$, matching of $\mathcal{S}_{M_2}(T_2)$ and $\mathcal{S}_L(T_1)$ follows from Lemma 2.7. (See also the remark following Lemma 2.7.)

Matching on (2) is similar to (1), albeit slightly more complicated to write down, so we omit details. □

4.2.3. $\mathbf{G} = E_7$.

**Proposition 4.3.** *As a* $G_2 \times \mathrm{GL}_3$*-module,* $r_{\bar{U}}(\mathbf{V})$ *has a filtration with three successive sub quotients*

(1) $i_{P_2\times Q_2}^{G_2\times \mathrm{GL}_3}(C_c^\infty(\mathrm{GL}_2))$.
(2) $i_{P_1\times Q_1}^{G_2\times \mathrm{GL}_3}(C_c^\infty(\mathrm{GL}_1)[-\frac{1}{2}, \frac{1}{2}])$.
(3) $\mathbf{V}(M) \oplus 1 \otimes |\det|$.

*Here* $\mathbf{V}(M)$ *is the minimal representation of* $M/C$.

This is a normalized version of Theorem 5.3 of [4] which states that $\mathbf{V}_{\bar{U}}$ has a filtration with three successive quotients

- $\mathrm{Ind}_{P_2\times Q_2}^{G_2\times \mathrm{GL}_3}(C_c^\infty(\mathrm{GL}_2)) \otimes |\det|^2$.
- $\mathrm{Ind}_{P_1\times Q_1}^{G_2\times \mathrm{GL}_3}(C_c^\infty(\mathrm{GL}_1)) \otimes |\det|^2$.
- $\mathbf{V}(M) \otimes |\det| \oplus 1 \otimes |\det|^2$.

*Proof of Theorem 1.1 in case* $\mathbf{G} = E_7$. The dual group of $H = \mathrm{PGSp}_6$ is $\mathrm{Spin}_7(\mathbb{C})$. Let $\mathbb{Z}^3$ be the root lattice of $\mathrm{Spin}_7(\mathbb{C})$ so that the short roots correspond to the standard basis

vectors $e_1, e_2, e_3$ in $\mathbb{Z}^3$. Let $V_8$ be the spin representation of $\mathrm{Spin}_7(\mathbb{C})$. The weights of $V_8$ are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Under the action of $\widehat{L} \cong \mathrm{GL}_3(\mathbb{C})$ the spin representation decomposes as

$$V_8 = V_1 \oplus V_3 \oplus V_3^* \oplus V_1^*$$

where $V_3$ is the standard representation of $\mathrm{GL}_3(\mathbb{C})$ and $V_1$ is the determinant character. The weights of these 4 summands are $(x, y, z) = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ such that $x+y+z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, respectively. We have an injection

$$r : G_2(\mathbb{C}) \to \mathrm{Spin}_7(\mathbb{C})$$

where $G_2(\mathbb{C})$ is defined as the stabilizer of a non-zero vector in $V_1$, for example. In particular, $G_2(\mathbb{C}) \cap \mathrm{GL}_3(\mathbb{C}) = \mathrm{SL}_3(\mathbb{C})$.

Let $V$ be a finite dimensional representation of $\mathrm{Spin}_7(\mathbb{C})$ and $T_1$ the corresponding Hecke operator for $\mathrm{PGSp}_6$. We restrict $V$ to $G_2(\mathbb{C})$, and let $T_2$ be the corresponding Hecke operator for $G_2$. We want to show that $T_2$ matches with $\mathcal{S}_L(T_1)$ on $r_{\bar{U}}(\mathbf{V})$. Matching on (3) in Proposition 4.3 trivially follows from the previously proved case $\mathbf{G} = E_6$.

To prove matching on (1) it suffices to show that $\mathcal{S}_{M_2}(T_2)$ matches with $\mathcal{S}_{L_2} \circ \mathcal{S}_L(T_1)$ on $C_c^\infty(\mathrm{GL}_2)$. We can assume that the dual group of $M_2$ is $\mathrm{GL}_{2,l}(\mathbb{C}) \subseteq \mathrm{SL}_3(\mathbb{C})$ where $g \in \mathrm{GL}_{2,l}(\mathbb{C})$ sits in $\mathrm{SL}_3(\mathbb{C})$ as a block diagonal matrix $(g, \det g^{-1})$. The operator $\mathcal{S}_{M_2}(T_2)$ corresponds to the restriction of $V$ to $\mathrm{GL}_{2,l}(\mathbb{C})$. On the other hand, $L_2 = \mathrm{GL}_2 \times \mathrm{GL}_1$ and $\mathcal{S}_{L_2} \circ \mathcal{S}_L(T_1)$ acts on $C_c^\infty(\mathrm{GL}_2)$ as the Hecke operator for $\mathrm{GL}_2$ that corresponds to the restriction of $V$ to the first factor of

$$\widehat{L}_2 = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \subseteq \mathrm{GL}_3(\mathbb{C}) \subseteq \mathrm{Spin}_7(\mathbb{C}).$$

The first factor of $\widehat{L}_2$ is conjugated to $\mathrm{GL}_{2,l}(\mathbb{C})$ in $\mathrm{Spin}_7$ by the reflection corresponding to the short root $e_3$. Matching on (1) now follows from Lemma 2.7.

To prove matching on (2) it suffices to show that $\mathcal{S}_{M_1}(T_2)$ matches with $\mathcal{S}_{L_1} \circ \mathcal{S}_L(T_1)$ on $C_c^\infty(\mathrm{GL}_1)$. We have $\widehat{M_1} \simeq \mathrm{GL}_{2,s}(\mathbb{C})$, where the center of $\mathrm{GL}_{2,s}(\mathbb{C})$ is the torus $\alpha_l^*(\mathbb{C}^\times) \subseteq \mathrm{SL}_{2,l}(\mathbb{C})$. By Lemma 2.6, $\mathcal{S}_{M_2}(T_2)$ acts on $C_c^\infty(\mathrm{GL}_1)$ as the Hecke operator for $\mathrm{GL}_1$ that corresponds to the restriction of $V$ to $\alpha_l^*(\mathbb{C}^\times)$, weighted by the eigenvalues of $\alpha_s^*(q^{1/2})$. On the other hand, $\mathcal{S}_{L_1} \circ \mathcal{S}_L(T_1)$ acts on $C_c^\infty(\mathrm{GL}_1)$ as the Hecke operator for $\mathrm{GL}_1$ that corresponds to the restriction of $V$ to $\mathrm{GL}_1(\mathbb{C})$, the first factor of

$$\widehat{L}_1 = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \subseteq \mathrm{GL}_3(\mathbb{C}) \subseteq \mathrm{Spin}_7(\mathbb{C}),$$

weighted by the eigenvalues of

$$s = \begin{pmatrix} q^{-\frac{1}{2}} & & \\ & q^{\frac{1}{2}} & \\ & & q^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & q^{\frac{1}{2}} & \\ & & q^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \subseteq \mathrm{Spin}_7(\mathbb{C}),$$

where the first matrix in the above product reflects the twisting $[-\frac{1}{2}, \frac{1}{2}]$ in (2), and the second comes from Lemma 2.5. The pairs $(\alpha_l^*(\mathbb{C}^\times), \alpha_s^*(q^{1/2}))$ (see (16)) and $(\mathrm{GL}_1(\mathbb{C}), s)$ are conjugated in $\mathrm{Spin}_7(\mathbb{C})$ by the reflection corresponding to the short root $e_3$. Matching on (2) now follows from Lemma 2.7. $\square$

4.2.4. $\mathbf{G} = E_8$. Let $Q_m = L_m U_m$ be the maximal parabolic subgroup of $L \cong \mathrm{GSp}_6$ (as in Section 3.2.1). Let $s, t$ be a pair of real numbers. For $m = 1, 2$, let $C_c^\infty(\mathrm{GL}_m)[s,t]$ be the vector space $C_c^\infty(\mathrm{GL}_m)$, with an $M_m \times L_m$-module structure defined by

$$(g_1, g_2) \cdot f(h) = |\det g_1|^s \, |i(g_2)|^t \, f(hg_1)$$

for any $(g_1, g_2) \in L_m \cong \mathrm{GL}_m \times \mathrm{GSp}_{6-2m}$ and by (22) for any $g \in M_m$.

**Proposition 4.4.** *As a $G_2 \times \mathrm{GSp}_6$-module, $r_{\bar{U}}(\mathbf{V})$ has a filtration with three successive sub quotients*

(1) $i_{P_2 \times Q_2}^{G_2 \times \mathrm{GSp}_6}(C_c^\infty(\mathrm{GL}_2)[1, -\frac{3}{2}])$.
(2) $i_{P_1 \times Q_1}^{G_2 \times \mathrm{GSp}_6}(C_c^\infty(\mathrm{GL}_1)[\frac{1}{2}, -\frac{1}{2}])$.
(3) $\mathbf{V}(M) \otimes |i|^{-1} \oplus 1 \otimes |i|$.

*Here $\mathbf{V}(M)$ is the minimal representation of $M/C$.*

The proof of this proposition is given in Section 5.

*Proof of Theorem 1.1 in case $\mathbf{G} = E_8$.* The map $r : G_2(\mathbb{C}) \to F_4(\mathbb{C})$ is described as follows. A split, simply connected group $G_{sc}$ of type $E_6$ can be realized as a subgroup of $\mathrm{GL}(J_{27})$ fixing $\det : J_{27} \to F$. As described in 3.2.2, there is a dual pair $G_2 \times SL_3 \subseteq G_{sc}$. The group $F_4$ is the subgroup of $G_{sc}$ consisting of elements fixing the identity matrix in $J_{27}$. It is easy to check that $(G_2 \times \mathrm{SL}_3) \cap F_4 = G_2 \times \mathrm{SO}_3$. This defines $r$, and the centralizer $C$ of $r(G_2(\mathbb{C}))$ is $\mathrm{SO}_3(\mathbb{C})$. The dual group $\widehat{L}$ of $L = \mathrm{GSp}_6$ is a Levi factor of type $B_3$. Let $i^* : \mathbb{C}^\times \to \widehat{L}$ be the co-character corresponding to the isogeny character $i$ of $\mathrm{GSp}_6$. Then $i^*(\mathbb{C}^\times)$ is the center of $\widehat{L}$. We can conjugate $G_2(\mathbb{C})$ in $F_4(\mathbb{C})$ so that

$$G_2(\mathbb{C}) \subseteq \mathrm{Spin}_7(\mathbb{C}) = [\widehat{L}, \widehat{L}].$$

In this way, $i^*(\mathbb{C}^\times)$ is a maximal torus in $\mathrm{SO}_3(\mathbb{C})$, the centralizer of $G_2(\mathbb{C})$. The image of $\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SO}_3(\mathbb{C})$ is $s = i^*(q)$.

Let $V$ be a finite dimensional representation of $F_4(\mathbb{C})$ and $T_1$ the corresponding Hecke operator for $F_4$. If $V = \sum V' \otimes V''$ is the restriction of $V$ to $G_2(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C})$, we have defined $T_2$ as corresponding to $\sum \mathrm{Tr}_{V''}(s) V'$. We want to show that $T_2$ matches with $\mathcal{S}_L(T_1)$ on $r_{\bar{U}}(\mathbf{V})$. The operator $\mathcal{S}_L(T_1)$ corresponds to the representation $\widehat{L}$ obtained by restricting $V$ to $\widehat{L}$. We now show matching on (3). Let $V = \sum V_n$ be the restriction of $V$ to $\mathrm{Spin}_7(\mathbb{C}) = [\widehat{L}, \widehat{L}]$, where $i^*(q)$ act as $q^n$ on $V_n$. Let $T$ be the Hecke operator for $\mathrm{PGSp}_6$ that corresponds to the representation $\sum q^{-n} V_n$ of $\mathrm{Spin}_7(\mathbb{C})$. Then, by Lemma 2.4, $\mathcal{S}_L(T_1)$ acts on $\mathbf{V}(M) \otimes |i|^{-1}$ as $T$ acts on $\mathbf{V}(M)$. Matching on $\mathbf{V}(M) \otimes |i|^{-1}$ now follows from the previously proved case $\mathbf{G} = E_7$. Let $s_p \in G_2(\mathbb{C})$ be the image of $\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ under the principal

$$f : \mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C}).$$

Then $T_2$ acts on $1 \otimes |i|$ as the scalar $\sum \mathrm{Tr}_{V''}(s) \cdot \mathrm{Tr}_{V'}(s_p)$. Under the inclusion $G_2(\mathbb{C}) \subseteq \mathrm{Spin}_7(\mathbb{C})$, the composite $f : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Spin}_7(\mathbb{C})$ is the principal $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{Spin}_7(\mathbb{C})$. By Lemma 2.4, $\mathcal{S}_L(T_1)$ acts on $1 \otimes |i|$ as the scalar $\sum q^n \mathrm{Tr}_{V_n}(s_p)$. Since

$$\sum q^n \, \mathrm{Tr}_{V_n}(s_p) = \sum \mathrm{Tr}_{V''}(s) \cdot \mathrm{Tr}_{V'}(s_p),$$

matching is now proved on (3). The remaining cases are similar to $\mathbf{G} = E_7$, so we leave them as an exercise. □

## 5. A Jacquet module for $E_8$

In this section we prove Proposition 4.4. In this case $N$ is a Heisenberg group. A starting point is the following (Theorem 6.1 [4]).

**Proposition 5.1.** *Let $\Omega$ be the $M$-orbit of the highest weight vector in $N/Z$. We have the following exact sequence of $\bar{P}$-modules,*

$$0 \to C_c^\infty(\Omega) \to \mathbf{V}_{\bar{Z}} \to \mathbf{V}_{\bar{N}} \to 0.$$

*The action of $\bar{P}$ on $f \in C_c^\infty(\Omega)$ is given by:*

- *For every $\bar{n} \in \bar{N}$*
$$\Pi(n)f(x) = \psi(\langle x, \bar{n}\rangle)f(x).$$
- *For every $m \in M$*
$$\Pi(m)f(x) = |i(m)|^5 f(m^{-1}xm).$$

*Here $\psi$ is a non-trivial additive character, $\langle\cdot,\cdot\rangle$ is a pairing between $N/Z$ and $\bar{N}/\bar{Z}$ induced by the Killing form, and $i : M \to \mathrm{GL}_1$ is the character obtained by acting on $\bar{Z}$. Moreover, $\mathbf{V}_{\bar{N}} \cong \mathbf{V}(M) \otimes |i|^3 \oplus |i|^5$ where $\mathbf{V}(M)$ is the minimal representation of $M$ with the center acting trivially.*

By Section 7 in [4] we have an identification of vector spaces

$$N/Z \cong \bar{N}/\bar{Z} \cong F \oplus J_{27} \oplus J_{27} \oplus F$$

so that the pairing $\langle\cdot,\cdot\rangle$ is given by

$$\langle (a, B, C, d), (a', B', C', d')\rangle = aa' + \mathrm{Tr}(BB') + \mathrm{Tr}(CC') + bb'.$$

Under these identifications, the action of $G_2$ on $N/Z$ is the obvious one, and the centralizer of $G_2$ in $\bar{N}/\bar{Z}$ is

$$\bar{U}/\bar{Z} \cong F \oplus J_6 \oplus J_6 \oplus F$$

where $J_6$ is the space of $3 \times 3$ matrices with coefficients in $F$. It follows that the orthogonal complement of $\bar{U}/\bar{Z}$ in $N/Z$ can be identified by $J^0 \oplus J^0$ where $J^0$ is the set of $B \in J$ of the form

$$B = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

with $x, y, z \in \mathbb{O}^0$. Given this, we may denote an element $(B, B') \in J^0 \oplus J^0$ by a six-tuple

$$(u, u') = ((x, y, z), (x', y', z'))$$

of traceless octonions. The action of $G_2 \times \mathrm{GSp}_6$ on these elements is simple to describe. First, $g \in G_2$ acts on every component of the six-tuple $(u, u')$ from the left, and $h \in \mathrm{GSp}_6$ acts by $(u, u')h^{-1}$. In this way, the character $i$ of $M$ restricts to the isogeny character of $\mathrm{GSp}_6$. We highlight the action of certain subgroups: $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \subseteq \mathrm{Sp}_6$ acting on the pairs $(x, x')$, $(y, y')$ and $(z, z')$ respectively in the obvious way, and $h \in \mathrm{GL}_3$ by

$$(uh^{-1}, u'h^t).$$

Note that $\mathrm{GSp}_6$ preserves, up to the isogeny character,

$$J^0 \oplus J^0 \to \wedge^2 \mathbb{O}^0 \qquad ((x, y, z), (x', y', z')) \mapsto x \wedge x' + y \wedge y' + z \wedge z'.$$

Let $\Omega_0$ be the intersection of $\Omega$ with $J^0 \oplus J^0$, the orthogonal complement of $\bar{U}/\bar{Z}$. It follows, from Proposition 5.1, that there is an exact sequence of $G_2 \times \mathrm{GSp}_6$-modules

$$0 \to C_c^\infty(\Omega_0) \to \mathbf{V}_{\bar{U}} \to \mathbf{V}_{\bar{N}} \to 0,$$

where $(g,h) \in G_2 \times \mathrm{GSp}_6$ acts on $f \in C_c^\infty(\Omega_0)$ by

$$\Pi(g,h)f(x) = |i(h)|^5 f(g^{-1}xh).$$

In order to understand $C_c^\infty(\Omega_0)$, we need to compute $G_2 \times \mathrm{GSp}_6$-orbits on $\Omega_0$.

**Proposition 5.2.** *The set $\Omega_0$ consists of pairs of $((x,y,z),(x',y',z')) \in J^0 \times J^0$ such that $\langle x,y,z,x',y',z'\rangle$ is a non-zero null subspace of $\mathbb{O}^0$, and such that*

$$x \wedge x' + y \wedge y' + z \wedge z' = 0.$$

*Moreover, $\Omega_0$ consists of two $G_2 \times \mathrm{GSp}_6$ orbits $\Omega_1$ and $\Omega_2$ where*

$$\Omega_m = \{((x,y,z),(x',y',z')) \in \Omega \mid \dim(\langle x,y,z,x',y',z'\rangle) = m\}.$$

*Proof.* If $(0,B,B',0) \in \Omega_0$ then, by Lemma 7.5 in [4], $B$ is a rank one matrix, $B^2 = \mathrm{Tr}(B)B$. Since $\mathrm{Tr}(B) = 0$, we have $B^2 = 0$, and this is equivalent to

$$x^2 = y^2 = z^2 = xy = yz = 0,$$

i.e. the entries of $B$ span a null subspace of $\mathbb{O}^0$. Acting by $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, we can replace $x$, $y$ and $z$ (all or some) by $x'$, $y'$ and $z'$, respectively. Hence $\langle x,y,z,x',y',z'\rangle$ is a non-zero null subspace of $\mathbb{O}^0$. If the dimension of this null space is 1 then, without loss of generality, we can assume that $x \neq 0$. Then $(u,u')$ is in the $\mathrm{GSp}_6$ orbit of $((x,0,0),(0,0,0))$. Since $G_2$ acts transitively on 1-dimensional null subspaces, we have one $G_2 \times \mathrm{GSp}_6$ orbit. If the dimension of the null space is 2 then, without loss of generality, we can assume that $x$ and $z$ are a basis of this space. Using the action of $\mathrm{GL}_3$ we can arrange that $y = 0$. Since

$$x' = ax + cz, y' = ex + fz, z' = bz + dx$$

for some $a,b,c,d,e,f \in F$,

$$x \wedge (ax + cz) + 0 \wedge (ex + fz) + z \wedge (bz + cx) = (c - d)(x \wedge z)$$

and this is 0 if and only if $c = d$. If $c = d$ then it is not too difficult to see that $(u,u')$ is in the $\mathrm{GSp}_6$ orbit of $((x,0,z),(0,0,0))$. Since $G_2$ acts transitively on 2-dimensional null subspaces, we have one $G_2 \times \mathrm{GSp}_6$ orbit. Thus, to finish the proof we must show that $c = d$. This is done in Lemma 5.3, using that $B' = A \times B$ (the cross product) for some $A \in J_{27}$, by Lemma 7.5 in [4]. $\square$

**Lemma 5.3.** *Suppose that $x, z \in \mathbb{O}^0$ be linearly independent such that $x^2 = z^2 = xz = 0$. Let $x_1, y_1, z_1 \in \mathbb{O}^0$, and set*

$$A = A_0 + \begin{pmatrix} 0 & z_1 & -y_1 \\ -z_1 & 0 & x_1 \\ y_1 & -x_1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & z & 0 \\ -z & 0 & x \\ 0 & -x & 0 \end{pmatrix}$$

*where $A_0 \in J_6$. If*

$$A \times B = \begin{pmatrix} 0 & z' & -y' \\ -z' & 0 & x' \\ y' & -x' & 0 \end{pmatrix}$$

*is such that $x', y', z' \in \mathbb{O}^0$ then $x', y', z' \in Fx + Fz$. Moreover,*

$$x' = bx + cz, \qquad \text{and} \qquad z' = az + cx \tag{24}$$

*for some constants $a, b, c \in F$.*

*Proof.* Since $G_2$ acts transitively on the set of 2-dimensional null spaces of $\mathbb{O}^0$, by the $G_2$ action (which commutes with the cross product), we may assume that $x = s_1$ and $z = t_2$, and let

$$\begin{aligned} x_1 &= a_1^x s_1 + a_2^x s_2 + a_3^x s_3 + b_1^x t_1 + b_2^x t_2 + b_3^x t_3 + c^x(s_4 - t_4), \\ y_1 &= a_1^y s_1 + a_2^y s_2 + a_3^y s_3 + b_1^y t_1 + b_2^y t_2 + b_3^y t_3 + c^y(s_4 - t_4), \\ z_1 &= a_1^z s_1 + a_2^z s_2 + a_3^z s_3 + b_1^z t_1 + b_2^z t_2 + b_3^z t_3 + c^z(s_4 - t_4) \end{aligned}$$

where the elements $s_i, t_j \in \mathbb{O}$ are the basis elements given in (14).

The cross product is given by

$$A \times B = A \circ B - \tfrac{1}{2} A \operatorname{Tr} B - \tfrac{1}{2} B \operatorname{Tr} A + \tfrac{1}{2}(\operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr}(A \circ B)).$$

From this a simple calculation shows that if $A = A_0$ then the condition (24) is satisfied[1]. We may therefore assume that $A_0 = 0$. We find that

$$A \times \left(\begin{smallmatrix} 0 & z & 0 \\ -z & 0 & x \\ 0 & -x & 0 \end{smallmatrix}\right) = \frac{1}{2} \begin{pmatrix} -\operatorname{Tr}(z_1 z) & y_1 x & z x_1 + z_1 x \\ x y_1 & 0 & z y_1 \\ x z_1 + x_1 z & y_1 z & -\operatorname{Tr}(x x_1) \end{pmatrix}.$$

Note that the fractor of $1/2$ can be safely ignored since it can be absorbed into $x_1, y_1, z_1$.

This imposes several conditions:

(A) $\operatorname{Tr}(z_1 z) = \operatorname{Tr}(x x_1) = 0$
(B) $x y_1, y_1 z \in \mathbb{O}^0$
(C) $x z_1 + x_1 z \in \mathbb{O}^0$

Notice that for $w = a_1 s_1 + a_2 s_2 + a_3 s_3 + b_1 t_1 + b_2 t_2 + b_3 t_3 + c(s_4 - t_4) \in \mathbb{O}^0$, we have

$$wz = wt_2 = a_2 s_4 + b_1 s_3 - b_3 s_1 - c t_2,$$

$$xw = s_1 w = -a_2 t_3 + a_3 t_2 + b_1 s_4 - c s_1.$$

Combining this calculation with condition (A) shows that $a_2^z = b_1^x = 0$. With (B) it implies that $a_2^y = b_1^y = 0$, and with condition (C) we get that $b_1^z = -a_2^x$. Putting this all together, we have

$$\begin{aligned} x' &= -y_1 z = b_3^y s_1 + c^y t_2, \\ y' &= x z_1 + x_1 z = (a_3^z - c^x) t_2 - (c^z + b_3^y) s_1, \\ z' &= -x y_1 = -a_3^y t_2 + c^y s_1. \end{aligned}$$

This proves the Lemma. $\square$

Proposition 5.2 implies that $C_c^\infty(\Omega_2)$ is a submodule of $C_c^\infty(\Omega_0)$ and $C_c^\infty(\Omega_1)$ is a quotient. Let $S_1$ and $S_2$ be the stabilizers of $((s_1, 0, 0), (0, 0, 0))$ and $((s_1, t_2, 0), (0, 0, 0))$ in $G_2 \times \mathrm{GSp}_6$, respectively. Let $P_m$ and $Q_m$, $m = 1, 2$, be the maximal parabolic subgroups in $G_2$ and $\mathrm{GSp}_6$, respectively, as introduced in 3.1.1 and 4.2.4. In particular, these parabolic groups come with maps $P_m \times Q_m \to \mathrm{GL}_m \times \mathrm{GL}_m$. Then $S_m$ is the inverse image of $\Delta(\mathrm{GL}(m))$, the diagonally embedded $\mathrm{GL}_m$ into $\mathrm{GL}_m \times \mathrm{GL}_m$. Now one can easily deduce that $\mathbf{V}_{\bar{U}}$, as a representation of $G_2 \times \mathrm{GSp}_6$, has the following three sub quotients, here induction is not normalized.

[1]This is the action of $\mathrm{Sp}_6$

(1) $C_c^\infty(\Omega_2) \cong \mathrm{Ind}_{P_2\times Q_2}^{G_2\times \mathrm{GSp}_6}(C_c^\infty(\mathrm{GL}_2)) \otimes |i|^5$.
(2) $C_c^\infty(\Omega_1) \cong \mathrm{Ind}_{P_1\times Q_1}^{G_2\times \mathrm{GSp}_6}(C_c^\infty(\mathrm{GL}_1)) \otimes |i|^5$.
(3) $\mathbf{V}_{\bar{N}} \cong \mathbf{V}(M) \otimes |i|^3 \oplus 1 \otimes |i|^5$.

Proposition 4.4 is simply a normalized version of this result.

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